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Automorphisms, Isotone self-maps and cycle-free orders

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Abstract

For cycle-free ordered sets, the ratio $|Aut(P)|/|End(P)|$ converges to zero as $|P|$ goes to infinity.

Although considerable attention has focussed, over the past two decades, on the fixed point property for ordered sets, little is known about the number of all isotone self-maps of an ordered set, or even about the ratio of the number of automorphisms to the number of all isotone self-maps. For an ordered set P let $End(P)$ stand for the set of all isotone maps, that is, all $f: P \to P$ such that $x \leqslant y$ implies $f(x) \leqslant f(y)$, and let $Aut(P)$ stand for all of its automorphisms, that is, $\alpha: P \to P$ which are one-to-one and onto, such that α and α^{-1} both are isotone, that is, belong to $End(P)$. In a recent article, Rival and Rutkowski [4] propose this conjecture.

Conjecture.

$$\lim_{|P| \to \infty} \frac{|Aut(P)|}{|End(P)|} = 0.$$

The conjecture is particularly striking in view of these two facts: almost every ordered set is 'rigid', that is, has only one automorphism, the identity (Prömel); every n-element ordered set has at least $2^{\sqrt{n}}$ isotone self-maps. Nevertheless the conjecture poses major challenges: enumerate $|Aut(P)|$ and $|End(P)|$. The only known lower bounds on $|End(P)|$ are derived by enumerating isotone self-maps onto chains which, alone, cannot be enough to settle the conjecture, for there are examples of ordered sets with more automorphisms than isotone self-maps onto chains. And yet, we know of

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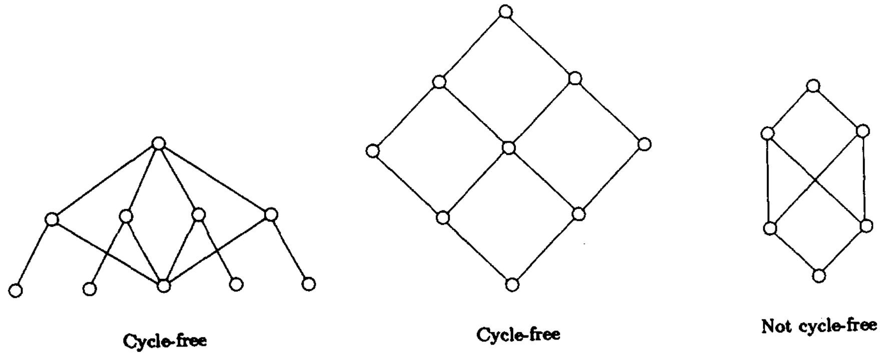

Fig. 1. Cycle-free and a non-cycle-free ordered sets.

no essentially different class of isotone self-maps of ordered sets — except those onto chains. Certainly the conjecture cannot be established using images of bounded size either, as even the n-element antichain illustrates. The purpose of this article is to make what seems a modest contribution to settling it.

Theorem 1. *For cycle-free ordered sets, the ratio* $|Aut(P)|/|End(P)|$ *converges to zero as* $|P|$ *goes to infinity.*

A subset of an ordered set P is a cycle if $x_i \leqslant y_i$, $x_{i+1} \leqslant y_i (\mathrm{mod}\ n)$ are the only comparabilities among these elements and, for the case $n = 2$, there is no element z in P such that $x_1 < z$, $x_2 < z$, $z < y_1$, $z < y_2$. An ordered set whose covering graph (that is, undirected diagram) contains no (graph) cycle is certainly cycle-free, as, for example, any ordered set whose covering graph is a tree (or forest), yet, there are cycle-free ordered sets which contain four-element subsets of type $a < b < d$, $a < c < d$ (see Fig. 1).

These ordered sets may have few, or many, automorphisms; for instance, a chain has only one, while an n-element antichain has $n!$ — both being cycle-free. In the process we shall deduce several striking aspects about the enumeration of the automorphisms of a cycle-free ordered set, for instance, the following.

Corollary 2. *In time* $\mathrm{O}(n^3)$, *we can, for a cycle-free ordered set* P, *compute* $|Aut(P)|$.

An element a *covers* an element b in an ordered set P if $a > b$ and if, for every c in P such that $a \geqslant c > b$, $a = c$. In this case a is an *upper cover* of b and b is a *lower cover* of a. An element a is *irreducible* in P if it has at most one upper cover and at most one lower cover. Let $I(P)$ stand for the subset of all irreducible elements of P. Consider the subdiagram of the diagram of the cycle-free ordered set P consisting of $I(P)$, that is, the directed induced subgraph with vertex set $I(P)$ of the diagram of P. This will be an ordered subset consisting of disjoint chains. Let $c(P)$ stand for the number of these chains.

Cycle-free ordered sets are closely related to dismantlable lattices, that is, lattices L which can be decomposed into the nested sequence of subsets $L_1 = L\backslash I(L)$, $L_2 = L_1\backslash I(L_1)$, $L_3 = L_2\backslash I(L_2), \ldots$ In a dismantlable lattice every irreducible element, apart from the top and bottom, has precisely one upper cover and precisely one lower cover.

In broad outline the proof of the main theorem will depend on establishing two inequalities. For any cycle-free ordered set P with at least six elements $|Aut(P)| \leqslant c(P)!$ and

$$|End(P)| \geqslant 2^{c(P)}|Aut(P)|.$$

Proof of Theorem 1. The starting idea for our main result is this observation: for any connected cycle-free ordered set P there are elements fixed by every automorphism. For instance, if P is a chain then P has just one automorphism, the identity, and that, of course, fixes every element. If P is not, then $\emptyset \neq I(P) \neq P$. Suppose that no member of $I(P)$ is a common fixed point of every automorphism of P. Obviously, for every α in $Aut(P)$, $\alpha(I(P)) = I(P)$.

On the other hand, for every $x \in I(P)$, there is α in $Aut(P)$ such that $\alpha(x) \neq x$. Now $P' = P\backslash I(P)$ is cycle-free, too, and every α in $Aut(P)$ satisfies $\alpha(P\backslash I(P)) = P\backslash I(P)$, that is $\alpha' = \alpha|_{P\backslash I(P)}$ is in $Aut(P')$. As a subset with the induced order, P' is still connected and so, by induction, $Aut(P')$ has a common fixed point which, in turn, is a common fixed point of $Aut(P)$. $\square$

Lemma 3. *Every automorphism of a cycle-free ordered set is determined by its action on the irreducible elements.*

Proof. Let P be a cycle-free ordered set and let $C_1, \ldots, C_k$ be the $c(P)$ chains of $I(P)$, that is, the chains of irreducible elements of P consecutively arranged with respect to the covering relation of P. For every i, we denote by $topC_i$, the element in P, if it exists, that covers the maximum element of C_i. We define, similarly, $bottomC_i$. Let α be any automorphism. Evidently α induces a permutation on the C_i's in which $\alpha(topC_i) = top\alpha(C_i)$ and, as $topC_i$has a unique upper cover in P, or none at all, so does $top\alpha(C_i)$, whence $top\alpha(C_i)$ is determined. Similarly, all $bottom(C_i)$ are determined, too. Finally, the complement P' of the C_i's in P is cycle-free and its irreducible elements are among the elements of $\{topC_i,\ bottomC_i\colon i = 1, 2, \ldots\}$ Applying induction completes the proof. $\square$

Our aim now is to show that for any cycle-free ordered set P,

$$|End(P)| \geqslant 2^{c(P)}|Aut(P)|.$$

Let $C_1, \ldots, C_k$ be the distinct covering chains of irreducible elements in P. We define two equivalence relations. The first, Θ_1, on these chains: $(C_i, C_j) \in \Theta_1$ if:

(i) $|C_i| = |C_j|$.

(ii) If $topC_i$ and $topC_j$ exist then $topC_i = topC_j$.

(iii) If $bottomC_i$ and $bottomC_j$ exist then $bottomC_i = bottomC_j$.

Let $a_1, a_2, \ldots$ be the sizes (number of chains) ranging over these Θ_1-equivalence classes. Next define Θ_2 on the set of Θ_1-classes:

$$([C_i]\Theta_1, [C_j]\Theta_1 \in \Theta_2$$

if, in P,

$$[C_i]\Theta_1 \cong [C_j]\Theta_1.$$

Let $b_1, b_2, \ldots$ be the sizes (number of Θ_1-classes) ranging over these Θ_2-classes.

Lemma 4. *For any cycle-free ordered set P,*

$$|Aut(P)| \leqslant c(P)!.$$

Proof. This follows from the observation that every automorphism induces a permutation of the chains of the Θ_1-classes, and, in view of Lemma 1, this action determines the automorphism. □

Lemma 5. *For any cycle-free ordered set P with $|P| \geqslant 6$,*

$$|End(P)| > 2^{c(P)}|Aut(P)|.$$

Proof. If P is an n-antichain, then

$$|End(P)| = n^n$$

and

$$|Aut(P)| = n!$$

from which the inequality readily follows, for $n \geqslant 6$. Let us suppose that P is not an antichain. We shall show, in particular, that

$$|End(P)| \geqslant \prod_i \frac{(a_i + 1)^{b_i}}{a_i!^{b_i}} |Aut(P)|.$$

Our aim is to show that $|Aut(P)|$ can be partitioned into classes each of size

$$r = \prod_i a_i!^{b_i}.$$

Let α be any automorphism of P. Then α induces a permutation on the Θ_1-classes of $I(P)$. Any permutation of the chains within a Θ_1-class and fixing $\alpha_{|P \setminus I(P)}$, produces an automorphism. As the permutations of the Θ_1-classes may be chosen independently there are r such automorphisms, all associated in this way with α, that is, agreeing with α on $P - I(P)$. Let β be any other automorphism, that is, an automorphism different from those manufactured above from α. Then according to Lemma 1, $\beta_{|P\backslash I(P)} \neq \alpha_{|(P)\backslash I(P)}$. As with α, we may now manufacture r more automorphisms, each associated with β (that is, identical to it on $P\backslash I(P)$). Continuing in this way, we may take γ different from all automorphisms so far constructed and manufacture r different ones, each agreeing with γ on $P\backslash I(P)$. If there are k such classes then $|Aut(P)| = rk$.

Next, with each such class of automorphisms we construct a corresponding class of isotone self-maps. Each such class of automorphisms may be represented by the chosen automorphism $\alpha, \beta, \gamma, \ldots$ The isotone self-maps that we construct shall agree with these automorphisms on $P\backslash I(P)$. In fact, we show that with each such class of r automorphisms there are

$$s = \prod_i (a_i + 1)^{a_i b_i}$$

isotone self-maps. Begin with $a \in |Aut(P)|$. For each Θ_1-class, say a_i chains of irreducibles with either *top* or *bottom*, there are $(a_i + 1)^{a_i}$ associated isotone self-maps, identical to α on the complement of this Θ_1-class since, for an antichain with a *top* (or *bottom*) of size a_i, there are $(a_i + 1)^{a_i}$ maps (all isotone) from the antichain into the antichain union the top (or the bottom). (Each of the a_i chains may be mapped, as one, that is, as a constant map of each chain.) Suppose each of these a_i chains in Θ_1-class is a singleton isolated from the rest of P. Then consider any element $x \in P$ and which is not in any of these chains. Clearly there are $(a_i + 1)^{a_i}$ maps (all isotone) from the antichain into the set union of the antichain and x. To complete the proof, we observe that, for each i,

$$\frac{(a_i + 1)^{a_i}}{a_i!} \geqslant 2^{a_i}$$

and, therefore,

$$\frac{s}{r} \geqslant 2^{\sum_i a_i b_i} = 2^{c(P)}.$$

Of course, an n-element ordered set contains a $\sqrt{n}$-element chain or a $\sqrt{n}$-element antichain. Suppose P has a $\sqrt{n}$-element chain K. For any chain $C = \{c_1 < c_2 < \cdots\}$ there is an isotone map f of P such that $f(P) = C$ ($f(x) = c_i$ for each $x \leqslant c_i$ and $x \nleqslant c_{i-1}$). Thus any subset of K is a chain corresponding to an isotone map of P, and there are $2^{\sqrt{n}}$ different subchains of K. Otherwise, P has an antichain of size $\sqrt{n}$. If P itself is an antichain then there are n^n set maps, all isotone. Otherwise, P contains a comparable pair, say $0 < 1$. Now, for each down set D of P the map which sends each member of D to 0 and the rest to 1, is isotone. Finally, a $\sqrt{n}$-element antichain gives rise to $2^{\sqrt{n}}$ antichains and, therefore, $2^{\sqrt{n}}$ down sets of P.

While counting $|Aut(P)|$ is surely a major obstacle to our principal conjecture, counting $|End(P)|$ is certainly a major one, too. Exploiting a little more, the idea of mapping onto chains, we can do somewhat better than $|End(P)| > 2^{\sqrt{n}}$. In fact, we can prove that $|End(P)| \geqslant 2^{cn}$, for some constant c (for example $c \geqslant 1/3$). □

Lemma 6. *For any ordered set P, $|End(P)| > 2^{n/3}$.*

Proof. Let P be an arbitrary n-element ordered set. If P has an antichain of size at least $n/3$ then $|End(P)| > 2^{n/3}$. Otherwise, P has a chain of size at least 3. Let $c_1 < c_2 < \cdots < c_m$ be such a chain and let

$$m_i = |\{x \in P\colon height(x) = i\}|,$$

where $height(x)$ stands for the size of a largest chain with $top = x$. Let $m_j = \min\{m_i\colon i = 1, 2, \ldots,\}$. Then $m_j \leqslant n/3$.

Now for every $i \neq j$, we consider any subset S_i of the set of elements of height i, and containing c_i, and let

$$S_j = \{x \in P\colon height(x) = j\}.$$

We can construct an isotone self-map as follows: $f(x) = c_i$ whenever $x \in C_i$, $i = 1, 2, \ldots$ Moreover, if $height(x) = i$ and $x \notin S_i$ then, for $i < j$, $f(x) = c_{i+1}$ and, for $i > j$, $f(x) = c_{i-1}$. It is routine to verify that f is isotone. The number of such maps is

$$\prod_{i \neq j} 2^{m_i - 1} = 2^{\sum_{i \neq j} m_i - 1} = 2^{n-(m-1)-m_j},$$

which is larger than $2^{n/3}$, unless $m > n/3$. Now an m-element chain has 2^m subchains and thus, if $m > n/3$, then $|End(P)| \geqslant 2^{n/3}$.

Recently, Duffus et al. [1] proved the better bound of $2^{2n/3}$. It should be possible to improve this bound to 2^n.

Finally, we are ready to complete the proof of the theorem. If

$$c(P) \geqslant \log n,$$

then according to Lemma 5 we are done. Otherwise, according to Lemma 4,

$$|Aut(P)| \leqslant \log n!.$$

Then from Lemma 6,

$$\frac{|Aut(P)|}{|End(P)|} \leqslant \frac{\log n!}{2^{n/3}},$$

which of course converges to zero as $|P|$ goes to infinity. □

Remarks. (1) The structure theory of cycle-free ordered sets may be used to efficiently enumerate $|Aut(P)|$. According to Lemma 1, each automorphism is determined by its

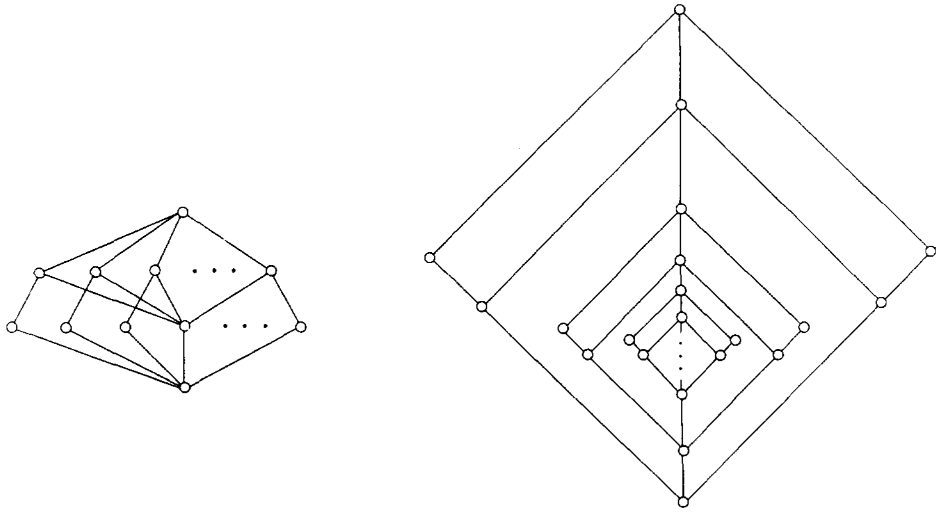

Fig. 2. Cycle-free ordered sets with many automorphisms.

effect on the irreducible elements $I(P)$. Of course, any automorphism of P determines an automorphism of $P\backslash I(P)$ and, indeed, according to the proof of Lemma 3, any automorphism of $P\backslash I(P)$ corresponds to

$$r = \prod_i a_i!^{b_i}$$

automorphisms of P, all identical on $P\backslash I(P)$. Thus

$$|Aut(P)| = |Aut(P\backslash I(P))| \prod_i a_i!^{b_i}.$$

A routine argument shows that the term

$$r = \prod_i a_i!^{b_i}$$

can be computed in time $O(n^2)$ (using, say, the matrix of covering relations of P), and this computation may be repeated up to n times as the cycle-free ordered set is 'dismantled', successively of its irreducible elements, that is, of $I(P)$, then of $P\backslash I(P)$, etc. This establishes the corollary.

(2) We do not, at this time, see how to extend our results to substantially wider classes of ordered sets.[3] Actually our main result's proof is based on the possibility to precisely count $|Aut(P)|$ for a cycle-free ordered set P. We know of few other cases for which the automorphisms can be computed.

[3] The essence of these ideas is used by Liu and Wan to establish this conjecture for some other classes of ordered sets, including, for instance, lattices.

On the other hand, the simple lower bound $2^{\sqrt{n}}$ for $|End(P)|$ is enough to complete the proof, although there are cycle-free ordered sets with more automorphisms (cf. Fig. 2).

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